# Solution of the Minimum Time-to-Climb Problem by Matched Asymptotic Expansions

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**Application of singular perturbation techniques to trajectory optimization problems of flight mechanics is discussed. The method of matched asymptotic expansions is used to obtain an approximate solution to the aircraft minimum time-to-climb problem. Outer, boundary-layer, and composite solutions are obtained to zeroth and first orders. A stability criterion is derived for the zeroth-order boundary-layer solutions (the theory requires a form of boundary-layer stability). A numerical example is considered for which it is shown that the stability criterion is satisfied and a useful numerical solution is obtained. The zeroth-order solution proves to be a poor approximation, but the first-order solution gives a good approximation for both the trajectory and the minimum time-to-climb. The computational cost of the singular perturbation solution is considerably less than that of a steepest descent solution. Thus singular perturbation methods appear to be promising for the solution of optimal control problems.**

## Nomenclature

$C_{D_0}$ = zero lift drag coefficient
$C_{L_\alpha}$ = lift curve slope
$D$ = drag divided by weight
$D'$ = drag, lb
$D_L$ = drag due to lift divided by weight
$D_0$ = zero lift drag divided by weight
$E$ = specific energy, sec$^2$
$F$ = thrust less zero lift drag, divided by weight
$h$ = altitude divided by acceleration due to gravity at sea level, sec$^2$
$h'$ = altitude, ft
$H$ = Hamiltonian function
$L$ = lift divided by weight
$L'$ = lift, lb
$M$ = Mach number
$Q$ = dynamic pressure times reference area, lb
$S$ = reference area, ft$^2$
$t$ = time, sec
$t_f$ = final time, sec
$t_f^*$ = minimum time-to-climb, sec
$T$ = thrust divided by weight
$T'$ = thrust, lb
$v$ = velocity divided by acceleration due to gravity at sea level, sec
$v'$ = velocity, fps
$w$ = airplane total weight, lb
$\alpha$ = angle of attack, deg
$\gamma$ = flight path angle, deg
$\eta$ = drag due to lift factor
$\lambda_i$ = adjoint variable associated with state variable $i$
$\rho$ = atmospheric density, slugs/ft$^3$

## Introduction

APPLICATION of the necessary conditions for optimal control of systems defined by ordinary differential equations results in a two-point boundary value problem. In many applications, including those involving atmospheric flight mechanics, the boundary-value problem is of sufficient complexity to require numerical solution. Although numerical solution of optimal control problems has received much attention in the past, all known methods exhibit deficiencies, most notably excessive computational cost.

There is at present no suitable method to compute optimal trajectories of atmospheric flight vehicles for preliminary design and performance estimation purposes. Methods based on highly simplified system models, such as "energy state," are easy to apply but exhibit undesirable features and may have considerable error. Methods using complex computer codes, most of which employ the method of steepest descent, solve the "exact" system but are usually tedious to apply and computationally expensive, and are therefore impractical for preliminary design.

The applicability and merits of a relatively new method of solution of optimal control problems are investigated in this paper. This method is based on the singular perturbation theory of ordinary differential equations and employs the technique of matched asymptotic expansions (MAE) to obtain solutions.

When confronted with a system of prohibitive computational complexity, one of the most logical and common approaches is to neglect terms in the equations which are thought to have only small effects on the solution. In the usual case, the approximate system has the same behavior as the original system. For example, consider the following initial value problem where $x$ is a scalar function and $\epsilon$ is a "small" scalar parameter:

$$\mathrm{d}x/\mathrm{d}t = f(x,t) + \epsilon g(x,t); \qquad x(\epsilon,0) = x_0$$

Under certain hypotheses, the solution of the system with $\epsilon = 0$ will give a good approximation to the solution of the original problem uniformly in the interval of interest; in particular, the initial condition can be met. This is termed a regular perturbation problem.

Now consider the system

$$\mathrm{d}x/\mathrm{d}t = f(x,y,t); \quad x(\epsilon,0) = x_0$$

$$\epsilon(\mathrm{d}y/\mathrm{d}t) = g(x,y,t); \quad y(\epsilon,0) = y_0$$

where $x$ and $y$ are scalar functions and $\epsilon > 0$ is a "small" scalar parameter. We call this the "exact" system and the system with $\epsilon$ set to zero the "reduced" system. It is obvious at once that in general the reduced solution will not be able to satisfy both initial conditions and thus, at least locally, the

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behavior of the reduced solution will be radically different from that of the exact solution. In fact, the best that can be hoped for is that the reduced solution gives a good approximation for $x$ uniformly in the domain of interest and for $y$ everywhere except near $y(\epsilon,0)=y_0$. This loss of boundary condition and consequent loss of uniform approximation is characteristic of singular perturbation problems. In spite of this radical change in solution behavior, singular perturbations are attractive because of the considerable simplification resulting from decreased system order.

Singular perturbation theory is concerned with the relation between the exact and reduced solutions of singularly perturbed systems of ordinary differential equations and with constructing asymptotic series representations of the exact solution. The fundamental results of this theory are due to Tihonov[1] and Vasileva.[2] Reviews of singular perturbation theory and methods are found in the books of Wasow[3] and O'Malley.[4] Unfortunately, most of the results of singular perturbation theory to date have been concerned with initial value problems whereas optimal control problems are of the two-point boundary value type. Applications of the theory to nonlinear optimal control problems has been studied most extensively by Kelley,[5,6] Hadlock,[7] and Ardema.[8] Applications to flight mechanics have been studied by Kelley,[5,9] Calise,[10-12] and Ardema.[8] The present paper is based on Ref. 8.

Largely independent of the development of singular perturbation theory for ordinary differential equations has been the development of asymptotic methods to solve certain fluid mechanics problems involving partial differential equations. These methods, most notably the method of matched asymptotic expansions (or method of inner and outer expansions), have their origin in the boundary-layer concept. In problems concerning viscous flow past a solid body, the viscosity is a parameter (usually small) multiplying the highest derivative in the Navier-Stokes equations. If this parameter is set equal to zero, the hydrodynamic system of equations results (reduced system); the solution of this system violates the no-slip boundary condition at the body surface. Thus, in a thin layer of fluid near the surface of the body—the boundary layer—the velocity varies rapidly from zero on the surface of the body to the value given by the hydrodynamic solution.

The phenomenon of boundary layers occurs in all singular perturbation problems. In such problems, the solution is sought in two (or in some cases, several) separate regions. In the outer region, the variables are relatively slowly varying, resemble the reduced solution, and do not in general satisfy boundary conditions. In the inner region near the boundary (boundary layer) the variables are relatively rapidly varying, asymptotically stable (hopefully), and satisfy appropriate boundary conditions.

A standard technique of obtaining approximate solutions of mathematical problems is to introduce perturbations about a nominal solution. This technique is particularly useful in problems in which a "small parameter" is present, because in this case the nominal solution and the method of introducing the perturbations are suggested in an obvious way. In some problems, no small parameter appears on physical grounds; such a parameter may be artificially inserted to suppress terms in the equation which are expected to have relatively small effects.

In the MAE method, separate solutions are obtained for the inner and outer regions by asymptotic expansion techniques. These asymptotic expansions need not be convergent and in fact often are not convergent in applications. The unknown constants are determined by "matching" the two solutions; the ability to do this depends on the existence of an overlap region of common validity. (Recall that the outer solution is not required to satisfy the boundary conditions.) If desired, the inner and outer solutions may then be combined to give a uniformly valid asymptotic representation of the solution. The MAE method has been developed by many researchers in fluid mechanics. Current expositions of the method are given by Cole,[13] Van Dyke,[14] and Nayfeh.[15] A more mathematical treatment is given by Eckhaus.[16]

In the MAE method it has been found to be advantageous not to give an explicit general formulation of the required expansions but to treat each new problem individually. This is due to the large diversity of behavior which may be encountered in even the most elementary problems, as well as to the great algebraic complexity generally involved. Experience has shown that it becomes readily apparent in the course of a solution whether or not the method is working.

One disadvantage of the MAE method is that it is not universally applicable. Common reasons for failure are: 1) there are no terms very much smaller than the others; in this case, the approximation will be poor; 2) the outer problem is trivial and the inner problem is identical to the exact one so that no simplification results; and 3) the boundary-layer equations are unstable which makes matching impossible.

## Zeroth-Order Solution

### Problem Formulation and Reduced Solution

A common formulation of the minimum time-to-climb (MTC) problem is as follows. We wish to minimize $t_f$ subject to the equations of motion

$$\dot{h}=v\sin\gamma,\quad \dot{E}=v(F-D_L),\quad \dot{\gamma}=(1/v)(L-\cos\gamma) \tag{1}$$

and the boundary conditions

$$h(0)=h_0,\quad h(t_f)=h_f$$

$$E(0)=E_0,\quad E(t_f)=E_f$$

$$\gamma(0)\ \text{either free or fixed}$$

$$\gamma(t_f)\ \text{either free or fixed}$$

A derivation of this formulation of the MTC problem and a discussion of the assumptions inherent in Eq. (1) are found in Ref. 17. Past experience indicates that among the state variables, $E$ is "slow" relative to $h$, and $h$ is "slow" relative to $\gamma$. It is this separation of the "speed" of the variables that motivates the selection of $E$ as a state variable.[8] Thus a possible singular perturbation formulation of the MTC problem is

$$\epsilon\dot{h}=v\sin\gamma,\quad \dot{E}=v(F-D_L),\quad \epsilon\dot{\gamma}=(1/v)(L-\cos\gamma) \tag{2}$$

The two reasons for the use of this formulation here are: 1) the associated reduced problem is the energy climb portion of the often used and easily obtained energy state approximation,[18,19] and 2) the other possible formulations result in problems with constrained control.

Applying the maximum principle for singularly perturbed systems[8] gives the necessary conditions as

$$\epsilon\dot{h}=v\sin\gamma \tag{3a}$$

$$\dot{E}=p(h,E,L) \tag{3b}$$

$$\epsilon\dot{\gamma}=(1/v)(L-\cos\gamma) \tag{3c}$$

$$\epsilon\dot{\gamma}_h=-\lambda_h\frac{\partial V}{\partial h}\sin\gamma-\lambda_E\frac{\partial p}{\partial h}+\lambda_\lambda\frac{1}{v^2}\frac{\partial v}{\partial h}(L-\cos\gamma) \tag{3d}$$

$$\dot{\lambda}_E=-\lambda_h\frac{\partial v}{\partial E}\sin\gamma-\lambda_E\frac{\partial p}{\partial E}+\lambda_\gamma\frac{1}{v^2}\frac{\partial v}{\partial E}(L-\cos\gamma) \tag{3e}$$

$$\epsilon\dot{\lambda}_\gamma=-\lambda_h v\cos\gamma-\lambda_\gamma(1/v)\sin\gamma \tag{3f}$$

$$-\lambda_E v(\partial D_L/\partial L)+\lambda_\gamma/v=0 \tag{4}$$

$$H=\lambda_0+\lambda_h v\sin\gamma+\lambda_E p+\lambda_\gamma (1/v)(L-\cos\gamma)=0 \tag{5}$$

where

$$p=v(F-D_L) \tag{6}$$

It is of interest to note that in this formulation the adjoint variable associated with the slow variable $E$ is itself slow and that the adjoint variables associated with the fast state variables are themselves fast.

The aerodynamics are assumed to be represented by

$$L=C_{L_\alpha}\alpha(1/w)Q$$

$$D=(C_{D_0}+\eta C_{L_\alpha}\alpha^2)(1/w)Q \tag{7}$$

where

$$Q=\tfrac{1}{2}\rho v'^2 S \tag{8}$$

We have

$$D_0=C_{D_0}Q/w \tag{9}$$

$$D_L=BwL^2 \tag{10}$$

where

$$B=\eta/(C_{L_\alpha}Q) \tag{11}$$

Using Eq. (10) in Eq. (4), the control law is

$$-2v^2\lambda_E BwL+\lambda_\gamma=0 \tag{12}$$

The zeroth-order outer (reduced) solution is obtained by setting $\epsilon=0$ in Eq. (3):

$$0=v_0^o\sin\gamma_0^o$$

$$dE_0^o/dt=p_0^o;\quad E_0^o(0)=E_0,\quad E_0^o(t_f)=E_f$$

$$0=(1/v_0^o)(L_0^o-\cos\gamma_0^o)$$

$$0=-\lambda_{h_0}^o\frac{\partial v_0^o}{\partial h_0^o}\sin\gamma_0^o-\lambda_{E_0}^o\frac{\partial p_0^o}{\partial h_0^o}+\lambda_{\gamma_0}^o\frac{1}{v_0^{o2}}\frac{\partial v_0^o}{\partial h_0^o}(L_0^o-\cos\gamma_0^o)$$

$$\frac{d\lambda_{E_0}^o}{dt}=-\lambda_{h_0}^o\frac{\partial v_0^o}{\partial E_0^o}\sin\gamma_0^o-\lambda_{E_0}^o\frac{\partial p_0^o}{\partial E_0^o}+\lambda_{\gamma_0}^o\frac{1}{v_0^{o2}}\frac{\partial v_0^o}{\partial E_0^o}(L_0^o-\cos\gamma_0^o)$$

$$0=-\lambda_{h_0}^o v_0^o\cos\gamma_0^o-\lambda_{\gamma_0}^o(1/v_0^o)\sin\gamma_0^o$$

The relations of Eqs. (12) and (5) lead to

$$-2v_0^{o2}\lambda_{E_0}^o B_0^o w+\lambda_{\gamma_0}^o=0$$

$$-1+\lambda_{E_0}^o p_0^o=0$$

where we have taken $\lambda_0=-1$ since $\lambda_0=0$ is not permissible (results in all adjoint variables identically equal to zero). Noting that

$$d\lambda_{E_0}^o/dt=-\lambda_{E_0}^o(\partial p_0^o/\partial E_0^o)$$

is redundant (since $H=0$ is a constant of the motion), the zeroth-order outer solution is given by

$$\gamma_0^o=0,\quad L_0^o=1,\quad \lambda_{h_0}^o=0 \tag{13a}$$

$$\partial p_0^o/\partial h_0^o=0,\quad dE_0^o/dt=p_0^o \tag{13b}$$

$$\lambda_{E_0}^o=1/p_0^o,\quad \lambda_{\gamma_0}^o=2v_0^{o2}\lambda_{E_0}^o B_0^o w \tag{13c}$$

This defines the energy climb path.[17-19] The energy state approximation consists of the energy climb path plus constant energy arcs which pass through the boundary conditions. These latter arcs do not appear in the singular perturbation treatment.

**Zeroth-Order Inner Solution**

The zeroth-order initial boundary-layer (inner) equations are formed by introducing the stretching transformation

$$\tau_0=t/\epsilon \tag{14}$$

into Eq. (3) and setting $\epsilon=0$. The result is

$$dh_0^{il}/d\tau_0=v_0^{il}\sin\gamma_0^{il} \tag{15a}$$

$$d\gamma_0^{il}/d\tau_0=(1/v_0^{il})(L_0^{il}-\cos\gamma_0^{il}) \tag{15b}$$

$$d\lambda_{h_0}^{il}/d\tau_0=(\lambda_{h_0}^{il}/v_0^{il})\sin\gamma_0^{il}-\lambda_{E_0}(\partial p_0^{il}/\partial h_0^{il})-(\lambda_{\gamma_0}^{il}/v_0^{il3})(L_0^{il}-\cos\gamma_0^{il}) \tag{15c}$$

$$d\lambda_{\gamma_0}^{il}/d\tau_0=-\lambda_{h_0}^{il}v_0^{il}\cos\gamma_0^{il}-(1/v_0^{il})\lambda_{\gamma_0}^{il}\sin\gamma_0^{il} \tag{15d}$$

where, from Eq. (12),

$$L_0^{il}=\lambda_{\gamma_0}^{il}/(2v_0^{il2}B_0^{il}w\lambda_{E_0}) \tag{16}$$

and where

$$v_0^{il}=[2(E_0-h_0^{il})]^{1/2} \tag{17}$$

and $\lambda_{E0}=\lambda_{E_0}^o(0)$ is a constant determined from Eq. (13). Similarly, the zeroth-order terminal boundary-layer equations are formed by introducing the stretching transformation

$$\tau_f=(t_f-t)/\epsilon \tag{18}$$

into Eq. (3) and setting $\epsilon=0$. The boundary conditions on Eq. (15) are $h_0^{il}(0)=h_0$ and either $\gamma_0^{il}(0)=\gamma_0$ or $\lambda_{\gamma_0}^{il}(0)=0$.

Asymptotic stability of the boundary-layer equations is of key theoretical (and computational) importance. The zeroth-order outer solution evaluated at the boundary is always an equilibrium (singular) point of the boundary-layer equations. By Theorem 4.1 of Ref. 8 we know that if 1) this equilibrium point is asymptotically stable, 2) the initial conditions are in the domain of influence of this root, and 3) the system functions are sufficiently smooth, then the reduced solution will be a "good" approximation to the full problem for $\epsilon$ sufficiently small, and better approximations may be obtained by asymptotic expansion techniques. A necessary condition for this stability is given by Theorem 5.2 of Ref. 8 for a simple control problem. The local stability characteristics of the boundary layer equations of the MTC problem are now investigated analytically.

To investigate the nature of the solutions of the boundary-layer equations in the vicinity of the zeroth-order outer solution set, using Eq. (13),

$$h_0^{il}=h_0^o+\delta_h;\quad \gamma_0^{il}=\delta_\gamma;\quad L_0^{il}=1+\delta_L \tag{19a}$$

$$\lambda_{h_0}^{il}=\delta_{\lambda h};\quad \lambda_{\gamma_0}^{il}=\lambda_{\gamma_0}^o+\delta_{\lambda\gamma} \tag{19b}$$

Substituting Eq.(19) in Eqs.(15) and (16) and eliminating $\delta_L$ gives

$$\frac{d\delta_h}{d\tau}=v\delta_\gamma \tag{20a}$$

$$\frac{d\delta_\gamma}{d\tau}=\frac{1}{v}\left(\frac{2}{v^2}-\frac{B_h}{B}\right)\delta_h+\frac{1}{v\lambda_\gamma}\delta_{\lambda\gamma} \tag{20b}$$

$$\frac{d\delta_{\lambda h}}{d\tau}=\left[\left(\frac{B_h}{B}-\frac{2}{v^2}\right)\left(\lambda_E p_{hL}+\frac{\lambda_\gamma}{v^3}\right)\right.$$
$$\left.-\lambda_E p_{hh}\right]\delta_h-\left(\frac{\lambda_E p_{hL}}{\lambda_\gamma}+\frac{1}{v^3}\right)\delta_{\lambda\gamma} \tag{20c}$$

$$\frac{d\delta_{\lambda\gamma}}{d\tau}=-\frac{\lambda_\gamma}{v}\delta_\gamma-v\delta_{\lambda h} \tag{20d}$$

where all super- and subscripts have been dropped for clarity and it is understood that all coefficients are to be evaluated on the zeroth order outer solution. This is a fourth-order linear system, to which the results of Chap. 3 of Ref. 8 apply.

According to Theorem 3.2 of Ref. 8, we desire two of the eigenvalues of the coefficient matrix of Eq. (20) to have positive real parts and two to have negative real parts. Then the two as yet free constants in the initial boundary layer may be chosen to suppress the unstable roots (associated with the eigenvalues that have positive real parts) in its solution, and the two as yet free constants in the terminal boundary layer may be chosen to suppress the unstable roots (associated with the eigenvalues which have negative real parts) in its solution.

The coefficient matrix for. Eq. (20) is

$$G=\left\{\begin{array}{cccc} 0 & v & 0 & 0 \\ \frac{1}{v}\left(\frac{2}{v^2}-\frac{B_h}{B}\right) & 0 & 0 & 1/v\lambda_\gamma \\ [\ ] & 0 & 0 & -\left(\frac{\lambda_E p_{hL}}{\lambda_\gamma}+\frac{1}{v^3}\right) \\ 0 & -\lambda_\gamma/v & -v & 0 \end{array}\right\} \tag{21}$$

The four eigenvalues $\omega_i$ of $G$ are the roots of

$$\omega^4+K_2\omega^2+K_1=0 \tag{22}$$

where

$$K_1=-P_{hh}/(2vBw),\quad K_2=(2B_h/B)-3/v^2 \tag{23}$$

We will have the case we want if the following criterion is satisfied:

either

$$4K_1>K_2^2 \tag{24a}$$

or

$$K_1>0 \text{ and } K_2<0 \tag{24b}$$

For physically meaningful problems, $K_1>0$; however, the other conditions of the criterion must be checked numerically for each specific problem.

We now turn to the integration of the initial boundary layer, Eq. (15). The relation of Eq. (5) is used to provide one of the two free initial conditions for Eq. (15). For the case of $\gamma(0)$ unspecified, the initial conditions are then

$$h_0^{il}(0)=h_0$$

$$\lambda_{\gamma_0}^{il}(0)=0$$

$$\lambda_{h_0}^{il}(0)=\frac{1-\lambda_{E_0}p_0^{il}(0)}{v_0^{il}(0)\sin\gamma_0^{il}(0)}$$

with $\gamma_0^{il}(0)$ selected to match with the outer solution as $\tau_0\rightarrow\infty$, i.e., to suppress the unstable modes. From Eq. (16) we note that $L_0^{il}(0)=0$.

To illustrate the solution a numerical example is now considered. The aircraft is "Airplane 2" of Ref. 19. The boundary conditions are selected as

$$h_0'=40{,}000\text{ ft};\quad M_0=0.5$$

$$h_f'=80{,}000\text{ ft};\quad M_f=2.0$$

Solving Eq. (13) gives the reduced (energy state) solution. Consider the initial boundary layer. The matching principle of the method of matched asymptotic expansions requires

$$\lim_{\tau_0\rightarrow\infty} h_0^{'il}(\tau_0)=h_0^{'0}(0)\approx 7500\text{ ft} \tag{25a}$$

$$\lim_{\tau_0\rightarrow\infty} \gamma_0^{il}(\tau_0)=\gamma_0^0(0)=0 \tag{25b}$$

$$\lim_{\tau_0\rightarrow\infty} \lambda_{h_0}^{il}(\tau_0)=\lambda_{h_0}^0(0)=0 \tag{25c}$$

$$\lim_{\tau_0\rightarrow\infty} \lambda_{\gamma_0}^{il}(\tau_0)=\lambda_{\gamma_0}^0(0)\approx 3.2 \tag{25d}$$

For this equilibrium point, $K_1=0.0000875$ sec$^{-4}$ and $K_2=0.00309$ sec$^{-2}$ so that Eq. (24a) is satisfied and the stability criterion is met. The time histories of $h_0^{'il}$ and $\gamma_0^{il}$ are shown in Figs. 1 and 2, respectively, for several values of the free initial condition $\gamma_0^{il}(0)$. As is to be expected in a numerical solution, there will always be at least a small portion of the unstable modes present, causing the solutions to diverge eventually. This is in contrast to the initial-value singular perturbation problem in which all modes are stable. The correct value of $\gamma_0^{il}(0)$ is that which satisfies Eq. (25). This is seen to be about $-68°$ or $-69°$. The time to reach the outer solution is about 52 sec. Determination of this time is not very sensitive to selection of $\gamma_0^{il}(0)$. The solution for the case of $\gamma(0)$ fixed, as well as the terminal boundary-layer solutions exhibit characteristics similar to those just discussed and will not be presented here.

### Zeroth-Order Composite Solution

For matching, the behavior of the boundary-layer solutions is needed as $\tau_0$, $\tau_f\rightarrow\infty$. Because small amounts of the unstable

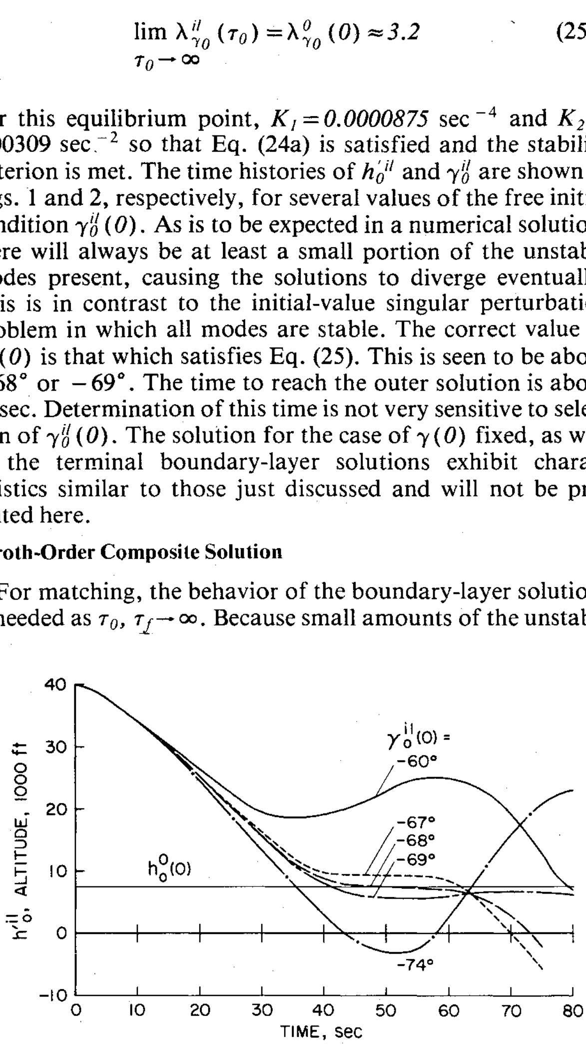


**Fig. 1 Behavior of $h'$ in the initial boundary layer, $\gamma_0^{il}(0)$ free.**

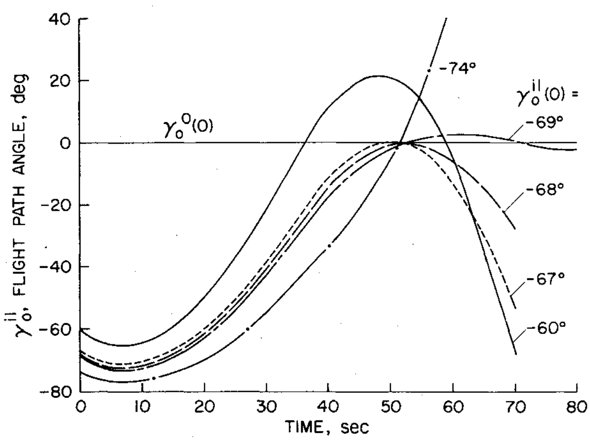


**Fig. 2 Behavior of $\gamma$ in the initial boundary layer, $\gamma_0^{il}(0)$ free.**

modes are inevitably present, it is natural to adopt the assumption that once the solution approaches equilibrium it will be held at equilibrium thereafter and thus stable numerical solutions are constructed.

Since there is one matching condition for each slow variable for each boundary layer, there are four matching conditions in the MTC problem which give the four unknown constants of integration.

For the zeroth order, the matching conditions are trivial and in fact have already been applied. The matching conditions on $E$ at the initial and final points give the boundary conditions for the reduced problem; namely, $E_0^0(0)=E_0$ and $E_0^0(t_f)=E_f$. The matching conditions on $\lambda_E$ give the values of $\lambda_E$ in the boundary layers; namely, $\lambda_{E_0}^0(0)$ in the initial layer and $\lambda_{E_0}^0(t_f)$ in the terminal one.

Collecting the outer solution and the two boundary-layer solutions gives a representation of the complete solution: a) near the initial point the solution is approximated by the initial boundary layer, b) away from the boundaries by the outer solution, and c) near the terminal point by the terminal boundary layer. This, however, is an awkward situation because it is not known a priori when one should "switch" from one representation to another. To circumvent this problem, composite solutions[13,14] are formed to give representations uniformly valid for $t\epsilon[0,t_f]$. Consider a variable $x$ with a boundary layer at $t=0$. The most common composite solution is the additive composition

$$x^a(\epsilon,t)=x^0(\epsilon,t)+x^i(\epsilon,t/\epsilon)-\mathrm{CP}_x(\epsilon,t)$$

where CP is the "common part," i.e., the terms which cancel out in the matching. The function $x^a$ has the properties that it satisfies the initial conditions, resembles the inner solution near $t=0$, resembles the outer solution away from $t=0$, is at least as accurate as each of its constituents, and is smooth. A common part will have to be subtracted out for each boundary layer present.

Since $E$ is constant in the boundary layers, $E_0^{i1}=\mathrm{CP}_{E_0}^{1}$ and $E_0^{i2}=\mathrm{CP}_{E_0}^{1}$, so that the zeroth-order additive composition for $E$ is

$$E_0^a(t)=E_0^0(t) \tag{26}$$

For $h$ we have

$$h_0^a(t)=h_0^0(t)+h_0^{i1}(t)+h_0^{i2}(t_f-t)-h_0^0(0)-h_0^0(t_f) \tag{27}$$

These functions are depicted in Fig. 3. For $\gamma$, $\gamma_0^0(t)=0$, so that

$$\gamma_0^a(t)=\gamma_0^{i1}(t)+\gamma_0^{i2}(t_f-t) \tag{28}$$

Other composite solutions are considered in Ref. 8.

The additive composition for the case of $\gamma$ free at the endpoints is compared with that of a selected problem with fixed end values of $\gamma$ on Fig. 4. Fixing $\gamma_0$ and $\gamma_f$ is seen to modify substantially the path. For the chosen values of $\gamma_0$ and $\gamma_f$, the boundary layers overlap so that at no point does the path lie on the energy climb path (outer solution). The times shown are obtained by adding the boundary-layer times to the energy state time.

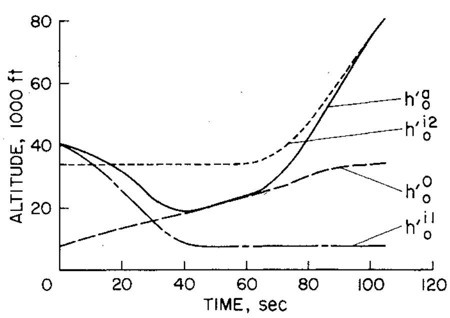


**Fig. 3 Zeroth-order additive composition of $h'$.**

## First-Order Solution

### First-Order Outer Solution

To improve upon the approximation of the zeroth order, the next higher order terms are now obtained. For the zeroth order terms, application of the method of MAE is essentially trivial in that the equations and the unknown constants may be determined without formally expanding the variables in $\epsilon$ and applying the matching principle. For the first and higher order terms, however, this is not the case and the procedures of the method must be carefully followed.

The outer problem is given by Eqs. (3), (5), and (12), the solution of which will be denoted by superscript 0. To solve these equations, the variables are expanded as power series in $\epsilon$ and only terms up to the first order are retained:

$$h^0=h_0^0+\epsilon h_1^0;\quad E^0=E_0^0+\epsilon E_1^0;\quad \gamma^0=\gamma_0^0+\epsilon\gamma_1^0$$

$$\lambda_h^0=\lambda_{h_0}^0+\epsilon\lambda_{h_1}^0;\quad \lambda_E^0=\lambda_{E_0}^0+\epsilon\lambda_{E_1}^0;\quad \lambda_\gamma^0=\lambda_{\gamma_0}^0+\epsilon\lambda_{\gamma_1}^0 \tag{29}$$

The coefficients in these series are functions of $t$ only. Now substitute Eq. (29) into Eqs. (3), (5), and (12), retain terms to the first order, and equate coefficients of like powers of $\epsilon$. Collecting the zeroth-order equations yields the zeroth-order outer solution of Eq. (13).

Collecting the first-order equations and using Eq. (13) yields the first-order outer problem:

$$\mathrm{d}h_0^0/\mathrm{d}t=v_0^0\gamma_1^0 \tag{30a}$$

$$\mathrm{d}E_1^0/\mathrm{d}t=p_1^0 \tag{30b}$$

$$\mathrm{d}\lambda_{\gamma_0}^0/\mathrm{d}t=-\lambda_{h_1}^0 v_0^0-\lambda_{\gamma_0}^0(1/v_0^0)\gamma_1^0 \tag{30c}$$

$$L_1^0=0;\quad 0=p_{h_1}^0 \tag{30d}$$

$$\mathrm{d}\lambda_{E_1}^0/dt=-\lambda_{E_1}^0 p_{E_0}^0-\lambda_{E_0}^0 p_{E_1}^0 \tag{30e}$$

$$2w[v_0^{0^2}\lambda_{E_0}^0(B_{h_0}^0 h_1^0+B_{E_0}^0 E_1^0)+v_0^{0^2}\lambda_{E_1}^0 B_0^0+2(E_1^0-h_1^0)\lambda_{E_0}^0 B_0^0]=\lambda_{\gamma_1}^0 \tag{30f}$$

$$0=\lambda_{E_0}^0 p_1^0+\lambda_{E_1}^0 p_0^0 \tag{30g}$$

In these equations, the zeroth-order outer variables (those with subscript zero) are known functions of time. Eq. (30b) may be integrated:

$$E_1^0=c_{E_1}^0\exp\int p_{E_0}^0\,\mathrm{d}t \tag{31}$$

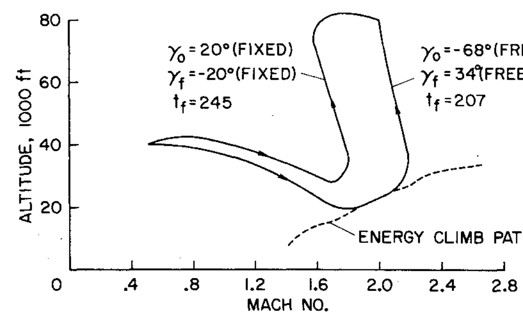


**Fig. 4 Comparison of zeroth-order additive composite paths for $\gamma_0$ and $\gamma_f$ free and fixed.**

The first-order outer solution may then be written as

$$\gamma_1^0 = (1/v_0^0)(\mathrm{d}h_0^0/\mathrm{d}t) \tag{32a}$$

$$L_1^0 = 0 \tag{32b}$$

$$\lambda_{h_1}^0 = -\frac{1}{v_0^0}\frac{\mathrm{d}\lambda_{\gamma 0}^0}{\mathrm{d}t} - \lambda_{\gamma 0}^0 \frac{1}{v_0^{0^2}}\gamma_1^0 \tag{32c}$$

$$E_1^0 = c_{E_1}^0 \exp\int p_{E_0}^0 \mathrm{d}t \tag{32d}$$

$$h_1^0 = \lambda_{E_0}^0 (\mathrm{d}h_0^0/\mathrm{d}t) E_1^0 \tag{32e}$$

$$\lambda_{E_1}^0 = -\lambda_{E_0}^{0^2} p_{E_0}^0 E_1^0 \tag{32f}$$

$$\lambda_{\gamma_1}^0 = 2w[v_0^{0^2}\lambda_{E_0}^0 (B_{h_0}^0 h_1^0 + B_{E_0}^0 E_1^0) + + v_0^{0^2}\lambda_{E_1}^0 B_0^0 + 2(E_1^0 - h_1^0)\lambda_{E_0}^0 B_0^0] \tag{32g}$$

where $c_{E_1}^0$ is a constant to be determined by the matching. It may be noted from these equations that $L=1$ and $\partial p(h, E, 1)/\partial h = 0$ are valid to first order. The first-order outer equations are linear and may be solved without recourse to numerical integration.

#### First-order Inner Solution

We now solve the initial boundary-layer (inner) problem for $E$ and $\lambda_E$ to first order. The solution of the terminal boundary-layer problem is similar and will not be discussed here. Introducing the transformation of Eq. (14) into Eqs. (3), (5), and (12) gives the initial boundary-layer equations for $E$ and $\lambda_E$ as

$$\frac{\mathrm{d}E^{il}}{\mathrm{d}\tau_0} = \epsilon p(h^{il}, E^{il}, L^{il}) = \epsilon p^{il} \quad \frac{\mathrm{d}\lambda_E^{il}}{\mathrm{d}\tau_0} = \epsilon\left[-\lambda_h^{il}\frac{1}{v^{il}}\sin\gamma^{il} - \lambda_E^{il}\frac{\partial p^{il}}{\partial E^{il}} + \lambda_\gamma^{il}\frac{1}{v^{il^3}}(L^{il} - \cos\gamma^{il})\right] \tag{33}$$

To solve these to first order in $\epsilon$, introduce

$$E^{il} = E_0^{il} + \epsilon E_1^{il}; \quad \lambda_E^{il} = \lambda_{E_0}^{il} + \epsilon\lambda_{E_1}^{il} \tag{34}$$

to obtain

$$\mathrm{d}E_0^{il}/\mathrm{d}\tau_0 = 0; \quad \mathrm{d}E_1^{il}/\mathrm{d}\tau_0 = p_0^{il} \tag{35}$$

$$\mathrm{d}\lambda_{E_0}^{il}/\mathrm{d}\tau_0 = 0 \tag{36a}$$

$$\mathrm{d}\lambda_{E_1}^{il}/\mathrm{d}\tau_0 = -(\lambda_{h_0}^{il}/v_0^{il})\sin\gamma_0^{il} - \lambda_{E_0}^{il} p_{E_0}^{il} + (\lambda_{\gamma_0}^{il}/v_0^{il^3})(L_0^{il} - \cos\gamma_0^{il}) \tag{36b}$$

The zeroth-order equations agree with those derived previously. The boundary conditions are

$$E_0^{il}(0) = E_0; \quad E_1^{il}(0) = 0 \tag{37}$$

Integrating,

$$E_1^{il}(\tau_0) = \int_0^{\tau_0} p_0^{il}(\eta)\,\mathrm{d}\eta \tag{38a}$$

$$\lambda_{E_1}^{il}(\tau_0) = c_{\lambda_{E_1}}^{il} + \int_0^{\tau_0} \frac{\mathrm{d}\lambda_{E_1}^{il}}{\mathrm{d}\tau_0}(\eta)\,\mathrm{d}\eta \tag{38b}$$

The remaining first-order equations are a fourth-order linear system with time-dependent coefficients. These equations are given in Ref. 8.

#### First-Order Composite Solution for Energy Variable and Final Integration

As discussed previously, there are four matching conditions for the MTC problem, one for each slow variable ($E$ and $\lambda_E$) at each boundary point. These are used to determine the remaining free constants in the inner and outer solutions. In addition, the minimum time-to-climb, $t_f^*$, is unknown.

The following procedure is now adopted: 1) the outer energy solution is matched with the initial boundary-layer solution and the constants of the outer solution, say $c_{E0}^{01}$ and $c_{E1}^{01}$, are determined; 2) an additive composite solution is formed, say $E_1^{a1}$. This solution is valid except in the region of influence of the terminal boundary layer; 3) the outer solution is matched with the terminal boundary-layer solution and the constants of the outer solution, say $c_{E0}^{02}$ and $c_{E1}^{02}$, are determined; 4) an additive composite solution is formed, say $E_1^{a2}$. This solution is valid except in the region of influence of the initial boundary layer; 5) $E_1^{a1}$ is propagated forward in time from the initial point and $E_1^{a2}$ is propagated backward from the terminal point until they meet in their region of mutual validity. This determines the minimum time-to-climb and gives a uniformly valid energy history. This procedure is necessary because we have a problem with free final time. It should be noted that the procedure would not work for nonautonomous systems.

We now match the outer solution with the initial boundary layer. The outer and initial inner solutions may be written as

$$E^0(\epsilon,t) = E_0^0(t) + \epsilon E_1^0(t) + \ldots \tag{39a}$$

$$E^{il}(\epsilon,\tau_0) = E_0^{il}(\tau_0) + \epsilon E_1^{il}(\tau_0) + \ldots \tag{39b}$$

respectively. The functions $E_0^0$, $E_1^0$, $E_0^{il}$, and $E_1^{il}$ are obtained from Eqs. (13), (32), (35), and (37). For matching, the behavior of the outer solution for small $t$ and of the inner for large $\tau_0$ is needed. For the former, the coefficients of Eq. (39a) are expanded in a power series about $t=0$. This gives

$$E^0(\epsilon,t) = E_0^0(0;c_{E_0}^{01}) + p_0^0(0)t + \epsilon c_{E_1}^{01} + \ldots \tag{40}$$

As before, to get the behavior of $E^{il}(\epsilon,\tau_0)$ for large $\tau_0$, the assumption is adopted that all boundary-layer variables achieve their equilibrium in a finite time, say $\tau_0^*$, and remain at equilibrium thereafter. Then for $\tau_0$ large (i.e., $>\tau_0^*$)

$$E_1^{il}(\tau_0) = p_0^{il}(\infty)\tau_0 - I_1^* \tag{41}$$

Thus,

$$E^{il}(\epsilon,\tau_0) = E_0 + \epsilon[p_0^{il}(\infty)\tau_0 - I_1^*] \tag{42}$$

where

$$I_1^* = p_0^{il}(\infty)\tau_0^* - \int_0^{\tau_0^*} p_0^{il}(\eta)\,\mathrm{d}\eta \tag{43}$$

The matching rule is

$$\lim_{\substack{\epsilon\to 0\\ t\to 0\\ \epsilon/t\to 0}}\left\{E^0(\epsilon,t) - E^{il}\left(\epsilon,\frac{t}{\epsilon}\right)\right\} = 0$$

Thus

$$\lim_{\substack{\epsilon\to 0\\ t\to 0\\ \epsilon/t\to 0}}\{E_0^0(0;c_{E_0}^{01}) + p_0^0(0)t + \epsilon c_{E_1}^{01} + \ldots - E_0 - p_0^{il}(\infty)t + \epsilon I_1^* - \ldots\} = 0 \tag{44}$$

Equating coefficients of like powers of $t^k \epsilon^e$ gives

$$E_0^0(0;c_{E_0}^{0I}) - E_0 = 0 \tag{45a}$$

$$p_0^0(0) - p_0^{iI}(\infty) = 0 \tag{45b}$$

$$c_{E_I}^{0I} + I_I^* = 0 \tag{45c}$$

The first of these gives the initial condition on $E_0^0$, which agrees with the value adopted previously. The validity of the second relation follows from the fact that the zeroth-order outer solution evaluated at $t=0$ is an equilibrium point of the initial zeroth-order inner solution and from the continuity of $p$. Equation (45c) gives the initial condition for $E_I^0$, namely,

$$E_I^0(0) = c_{E_I}^{0I} = -I_I^* \tag{46}$$

Then

$$E_I^0(t) = -I_I^* \exp\left[\int_0^t p_{E_0}^0(\eta)\,\mathrm{d}\eta\right] \tag{47}$$

All constants of integration have now been determined.

To form the initial additive composite, $E_I^{aI}$, note from Eqs. (44) and (45) that the common part is

$$CP_{E_I}^I(\epsilon,t) = E_0 + p_0^{iI}(\infty)t - \epsilon I_I^* \tag{48}$$

The function $E_I^{aI}$ is now formed according to

$$E_I^{aI}(\epsilon,t) = E_0^0(t) + \epsilon E_I^0(t)$$

$$+ E_0^{iI}(t/\epsilon) + \epsilon E_I^{iI}(t/\epsilon) - CP_{E_I}^I(\epsilon,t)$$

or, setting $\epsilon$ to its correct value, $\epsilon = 1$,

$$E_I^{aI}(1,t) = E_0^0(t) - I_I^* \exp\left[\int_0^t p_{E_0}^0(\eta)\,\mathrm{d}\eta\right]$$

$$+ \int_0^t p_0^{iI}(\eta)\,\mathrm{d}\eta - p_0^{iI}(\infty)t + I_I^* \tag{49}$$

The function $E_I^{aI}$ has been computed for the numerical problem defined earlier. The results are shown in Fig. 5. It is seen that inclusion of the first-order terms has resulted essentially in a delay in energy accumulation. The boundary layer influences about the first 60 sec of the solution. The time to get to a specified level of energy has increased by aboiut 33 sec compared to the zeroth-order solution.

The matching of the outer solution with the terminal boundary layer and formation of $E_0^{a2}$ is similar to the procedure for $E_I^{aI}$ and will not be given here. We now combine $E_I^{aI}$ and $E_I^{a2}$ to obtain a uniform energy history, $E_I^a$, and improved estimate of the minimum time-to-climb, $t_f^*$. To do this, $E_I^{aI}$ is propagated forward in time from $t=0$ and $E_I^{a2}$ backward from $t_f$ until they meet in a region of common validity. The result is shown in Fig. 6 for a specific problem. The resulting $t_f^*$ is 162.4 sec.

To complete the solution to first order in all variables, the following two procedures suggest themselves. 1) Solve the first-order boundary-layer systems for both the initial and terminal layers to get the first-order inner solutions for $h$ and $\gamma$. Then form composite solutions in the same manner as was done for $E$. This procedure would give solutions that meet all boundary conditions but which would not satisfy the state equations exactly. 2) Use the first-order additive composite solution for $E$ to integrate the exact state equations. This gives solutions that satisfy the state equations but that would not meet all the terminal boundary conditions exactly.

The second of these procedures is adopted here; the original state equations, Eq. (1), are solved subject to the specified initial conditions. In these equations, we take $E = E_I^a$. Since $E$ is a known function, $\mathrm{d}E/\mathrm{d}t = \dot{E}_I^a$ is also known. The results of the integration are presented in Figs. 7 and 8. To provide a solution of known validity for comparison purposes, a steepest descent solution of the example was obtained using the computer program of Ref. 20. The steepest descent solution is shown in the figures for comparison. The comparison shows that there is excellent agreement for all

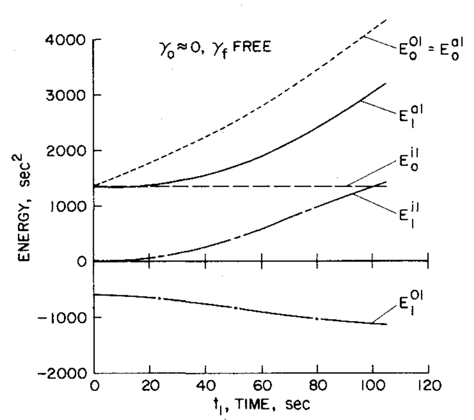

**Fig. 5 Energy time histories matched at initial point.**

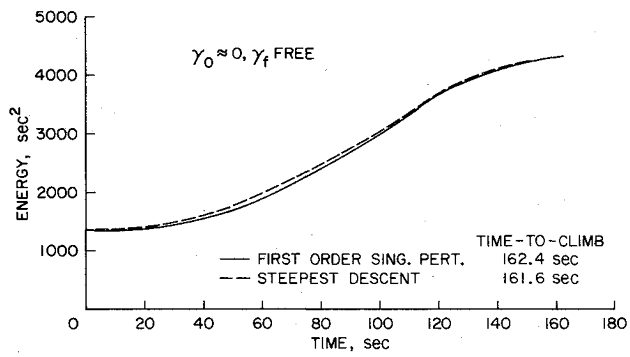

**Fig. 6 Comparison of energy time histories.**

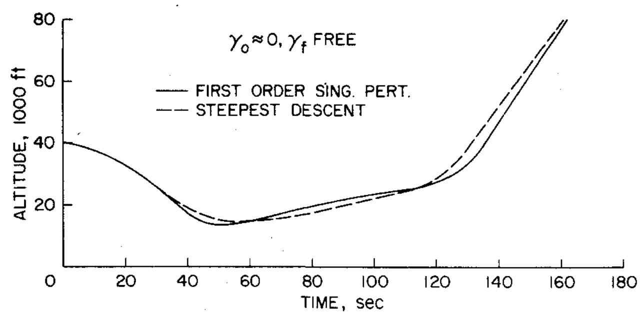

**Fig. 7 Comparison of altitude time histories.**

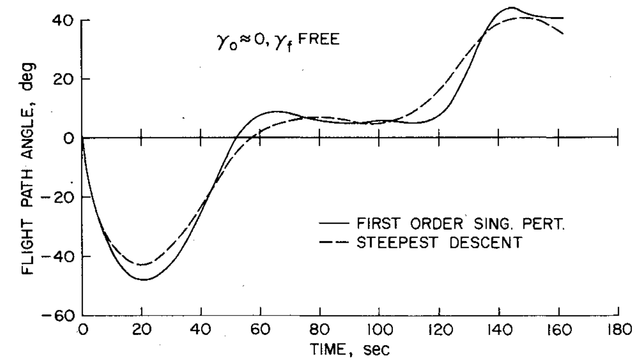

**Fig. 8 Comparison of flight path angle time histories.**

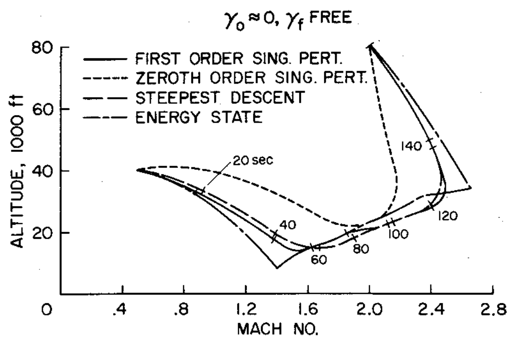


Fig. 9 Comparison of paths.

**Table 1 Comparison of minimum time-to-clumb by various methods**

| Method | Time-to-Climb[a] |
|---|---|
| Energy state | 105 |
| Zeroth order singular perturbation[b] | 214 |
| First-order singular perturbation[c] | 162 |
| Steepest descent | 162 |

[a] $h_0' = 40{,}000$ ft, $h_f' = 80{,}000$ ft, $M_0 = 0.5$, $M_f = 2.0$, $\gamma_0 \approx 0$, $\gamma_f$FREE.
[b] Boundary layer times added to energy state values.
[c] Additive composition.

variables. However, the slower variables tend to give better agreement with the steepest descent solution than do the faster ones.

The paths of the various solutions in the $(h', M)$ plane are shown in Fig. 9 with the corresponding estimates of the minimum time-to-climb listed in Table 1. Figure 9 shows a substantial discrepancy between the steepest descent path and the energy state and zeroth-order singular perturbation paths. However, there is excellent agreement between the steepest descent and the first-order singular perturbation paths. The energy state and zeroth-order solutions underestimate and overestimate, respectively, the minimum time-to-climb by substantial amounts, as shown in the table. On the other hand, the minimum times-to-climb as predicted by the steepest descent and first-order solutions differ by about ½%. The first-order singular perturbation solution required about 1/40 of the computational cost of the steepest descent solution.

## Conclusions

Based on application to a flight trajectory optimization problem (minimum time-to-climb), singular perturbation methods appear attractive for solution of optimal control problems. Specifically, an approximate solution of the minimum time-to-climb problem was obtained by the method of matched asymptotic expansions. The solution was in excellent agreement with a solution obtained by an established method, but required substantially less computational cost. There are some limitations to the use of singular perturbation methods, the most serious of which may be that the general characteristics of system behavior must be known a priori. However, when such methods are applicable, they are likely to give good results with a minimum amount of computational cost.

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